113 Class Problems: The Orbit-Stabilizer Theorem

1. Let a group G act on a set S. Prove that $Stab(x) \subset G$ is a subgroup.

   Solution:

- $e(x) = x \Rightarrow e \in Stab(x)$
- $f, g \in Stab(x) \Rightarrow (f \circ g)(x) = f(g(x)) = f(x) = x \Rightarrow f \circ g \in Stab(x)$
- $g \in Stab(x) \Rightarrow g(x) = x \Rightarrow g^{-1}(x) = g^{-1}(g(x)) = (g^{-1} \circ g)(x) = e(x) = x$
  $\Rightarrow g^{-1} \in Stab(x)$

2. Let a group G act on a set S, where $|G| = 35$ and $|S| = 4$. Prove that the action is trivial. An action is trivial if $g(x) = x$ for all $x \in S$ and $g \in G$.

The action is trivial $\Leftrightarrow |Orb(x)| = 1 \quad \forall x \in S$

$Orb(x) \subset S \Rightarrow |Orb(x)|$ could potentially be $1, 2, 3$ or 4

Orbit-Stabilizer $\Rightarrow |Orb(x)| \mid |G| \Rightarrow |Orb(x)| \mid 35$

$\Rightarrow |Orb(x)| = 1 \quad \forall x \in S$

$\Rightarrow$ Action is trivial.